Linear pressure waves in fogs

By R. DURAISWAMI[1] AND A. PROSPERETTI[2]

[1]Dynaflow Inc., 7210 Pindell School Road, Fulton, MD 20759, USA

[2]Department of Mechanical Engineering, The Johns Hopkins University,
Baltimore, MD 21218, USA

The modelling of small-amplitude pressure waves in dilute single- or multi-component fogs by means of averaged equations is considered. The problem is cast in a singular-perturbation framework in which the suspended droplets are the singularities. This point of view simplifies the local problem in the vicinity of the droplets. Matching in the overlap region provides the coupling with the averaged fields. Among the advantages of the method is the fact that the leading-order effects are clearly identified. In particular it is shown that, for low-amplitude waves and far below the fluid's critical point, phase change effects only start to be important when the vapour mean free path becomes comparable with the drop radius and dominate for yet smaller drops.

The present method for the derivation of effective equations appears to be of general applicability to a variety of multi-phase situations and is illustrated in detail.

1. Introduction

The propagation of pressure waves in a suspension of small liquid droplets in a gas or gas–vapour mixture is a complex process of obvious interest in science and technology. Like so many other multi-phase processes, its theoretical description is largely based on equations that have been postulated on the basis of physical arguments rather than derived in a systematic manner. It is the purpose of the present contribution to provide the derivation of a continuum model applicable to the case of linear waves in the dilute case.

Superficially, the averaged equations that we derive resemble many others available in the literature. However our model is not heuristic, but has a strong mathematical basis. Secondly, our approach clarifies in a systematic way what are the important effects to leading order, a task that is not within easy reach of heuristic arguments. We thus not only gain valuable insight into the controlling physical processes, but also derive the simplest model which is consistent to leading order in the droplet volume fraction. In particular we show that, for weak waves and below the critical point of the fluid, phase-change effects start being important only when the Knudsen number Kn is of order 1, i.e. when the drop radius becomes comparable with the mean free path. For bigger drops energy exchanges are too weak to give rise to appreciable evaporation and condensation. The range of $Kn \sim 1$ is of particular interest because the usual continuum laws break down. The experimental study of linear waves offers therefore a tool to investigate the physical nature of processes in this regime. In our theoretical work we have to rely on semi-empirical information (Gyarmathy 1982) to find explicit expressions for the inter-phase coupling terms, i.e. the drop evaporation

rate, drag, and heat transfer. The explicit forms that we derive for these quantities are significant results of the present work.

The third point of interest of this study is a technical one. We further develop Foldy's (1945) technique as modified by Caflisch et al. (1985) for the study of the propagation of pressure waves in bubbly liquids. Since this technique is simple, powerful, and of general applicability, it is useful to set it out systematically in general terms, which we do in §4. The basic idea is to consider the inhomogeneities (in the present case, the droplets) as providing a singular perturbation to the field equations for the gas. The problem in the droplets' near-field – the inner region – can then be solved subject to matching to the averaged fields in the overlap domain. The advantage is a clear decoupling between the complexity of the near-field behaviour and the dominant effects at a distance from the drops. Although it turns out that, for a fog with strong phase-change effects, due to the large value of the latent heat, the problem that we can handle with this technique is essentially linear, in other situations nonlinear effects originating near the inhomogeneities appear in the averaged equations as in the case treated by Caflisch et al.

The literature on the propagation of pressure waves in fogs is quite extensive and we make no attempt to provide an exhaustive list of references. A separate (and vast) literature is also available on pressure waves in suspensions and emulsions, and on the flow of particle-laden mixtures. The most transparent physical justification of the mathematical models used in the literature on fogs and wet vapours is perhaps that provided by Marble (1969) and Marble & Wooten (1970). Wei & Wu (1981) have reviewed the literature up to about 1980. A synthesis of Soviet work in this area has been provided by Gumerov, Ivandaev & Nigmatulin (1988). Young (1982, 1984) studied the flow of wet steam in turbines and later, with Guha, the structure of shock waves propagating in vapour–droplet mixtures (Young & Guha 1991; Guha 1992, 1994). Finite Knudsen number effects for sound absorption in fogs have been studied by Fukuta & Walter (1970), Jaeschke, Heller & Meier (1975), and Wei, Tian & Lu (1987). A complete list of references that traces the history of the problem from the 19th century can easily be constructed from these works.

2. Problem statement

We consider a gas–vapour mixture, satisfying the perfect-gas laws, which we simply refer to as the vapour. We use the standard conservation laws for mass

$$\nabla \cdot \boldsymbol{u} = -\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t}, \tag{2.1}$$

momentum

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p - \mu \nabla^2 \boldsymbol{u} = -\eta \mu \nabla (\nabla \cdot \boldsymbol{u}), \tag{2.2}$$

and energy

$$\rho \, C_p \frac{\mathrm{d}T}{\mathrm{d}t} - k \nabla^2 T - \boldsymbol{\tau} : \boldsymbol{e} = \frac{\mathrm{d}p}{\mathrm{d}t}. \tag{2.3}$$

Here ρ is the density, p the pressure, T the temperature, $\boldsymbol{u}$ the velocity, μ the shear viscosity, $\eta \mu$ the bulk, or second, viscosity, C_p the specific heat, k the thermal conductivity, $\boldsymbol{\tau}$ the viscous stress tensor, and $\boldsymbol{e}$ the rate of deformation tensor; $\mathrm{d}/\mathrm{d}t$ denotes the convective derivative. Compressibility of the liquid phase is neglected and the appropriate equations can therefore be obtained from the previous ones by

simply setting all the right-hand sides to zero. The subscript L will be used for the liquid quantities, while no subscript is attached to the vapour ones.

In the presence of an incondensible gas, (2.1) to (2.3) remain valid except that ρ, C_p, and K denote the total mixture density, specific heat, and conductivity, and u the centre-of-mass velocity. (A nonlinear correction to the energy equation will not contribute in the following.) One also needs the diffusion equation

$$\rho \frac{dc}{dt} = \nabla \cdot (\rho \mathscr{D} \nabla c) \,, \tag{2.4}$$

where $\mathscr{D}$ is the diffusivity of mass and c the mass fraction of the vapour phase. Thermal and pressure diffusion have been neglected. The treatment of diffusion effects is deferred until §7. Until then, we shall consider the pure-vapour case only.

As mentioned before, in the problem studied here the mean free path of the vapour molecules is not necessarily small compared with the droplet radius. For this reason some of the previous equations need to be amended in the neighbourhood of the drops. These issues are dealt with in §5.

At the liquid–vapour interfaces conservation of mass requires that

$$\rho\,(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{n} = \rho_L(\boldsymbol{u}_L - \boldsymbol{w}) \cdot \boldsymbol{n} \equiv \dot{m}, \tag{2.5}$$

where $\boldsymbol{n}$ is the unit normal directed out of the drop and $\boldsymbol{w}$ is the velocity of the interface. The mass flux per unit area $\dot{m}$ as defined by this relation is positive for evaporation. The incondensible gas will be assumed to be insoluble in the liquid and therefore $\dot{m}$ is entirely due to the vapour flow in the multi-component case. Explicit expressions for the normal and tangential momentum boundary conditions are not required in the following.

Conservation of energy is expressed by

$$- (k_L \nabla T_L + \boldsymbol{q}) \cdot \boldsymbol{n} = L\dot{m}, \tag{2.6}$$

where L is the enthalpy of vaporization and $\boldsymbol{q}$ the heat flux on the vapour side. Owing to the finiteness of the Knudsen number the relation between $\boldsymbol{q}$ and T may be expected to differ from Fourier's law applicable in the bulk of the vapour.

In the presence of evaporation and heat transfer the temperature field undergoes a jump in traversing the interface from the vapour to the liquid (Pao 1971a,b; Kogan 1973; Gyarmathy 1982; Labuntsov & Kryukov 1979; Cercignani 1988; Onishi 1986; Aoki, Sone & Yameda 1990; Sugimoto & Sone 1992). Kinetic theory gives a relation of the form (see e.g. Labuntsov & Kryukov 1979)

$$\frac{T_s - T_{ex}}{T_s} = \beta_1 \frac{\dot{m}}{\rho_v^e(T_s)\,(2\mathscr{R}_v T_s)^{1/2}} + \beta_2 \frac{\boldsymbol{q} \cdot \boldsymbol{n}}{p^e(T_s)\,(2\mathscr{R}_v T_s)^{1/2}} \,, \tag{2.7}$$

where β_1, β_2 are numerical coefficients of order 1, T_s is the surface temperature, T_{ex} is the vapour temperature extrapolated to the surface, $\rho_v^e(T_s)$ and $p_v^e(T_s)$ are the equilibrium saturation density and pressure, and

$$\mathscr{R}_v = \frac{R_G}{M_v}, \tag{2.8}$$

with R_G the universal gas constant and M_v the vapour molecular weight. The first term in the right-hand side of (2.7) is proportional to the Mach number of the phase change process at the interface. As for the second term, it is easy to show that it is of the order of the temperature change over a mean free path divided by the absolute temperature. Both effects can be expected to be small in the present application in

which the equations are linearized and will be neglected. Hence, at the interface, we assume that

$$T = T_L = T_s. \tag{2.9}$$

One further relation expressing the vapour conditions at the interface is necessary. This is a rather complex and unsettled issue at present (see e.g. Aoki & Cercignani, 1983; Ytrehus, 1983; Koffman, Plesset & Lees, 1984; Cercignani, Fiszdon & Frezzotti 1985; Hermans & Beenakker 1986; Onishi 1986; Shankar & Deshpande 1990). To explore the sensitivity of the results to this aspect of the model we shall make use of two different conditions, namely saturation, and a relation of the Hertz–Knudsen form

$$\dot{m} = \alpha \left[\frac{\mathscr{R}_v T_s}{2\pi} \right]^{1/2} \left[\rho_v^e(T_s) - \rho \right] . \tag{2.10}$$

Here α would equal the accommodation coefficient α_c for the original Hertz–Knudsen relation. Later studies suggest that setting $\alpha = 2\alpha_c/(2 - \alpha_c)$ may be more accurate (see e.g. Kogan 1973).

In what follows we shall treat in detail the case of a fog consisting only of the liquid and its vapour. The presence of an incondensible gas only introduces slight differences and is considered in §7. We shall also assume all drops to have equal radius. As noted at the end of §4, the generalization to a size distribution is straightforward.

3. Scaling and linearization

The method to be used in the derivation of averaged equations is essentially a method of singular perturbations and requires separate scaling analyses at the macroscopic, or average, level and at the microscopic, or local, level (i.e. in the neighbourhood of each droplet). The idea is sketched in figure 1.

3.1. Macroscopic scaling

From the macroscopic point of view, the passage of a pressure wave through the fog is characterized by a typical wavelength λ and frequency ω in terms of which we non-dimensionalize lengths and times,

$$x = \lambda x_*, \qquad t_* = \omega t. \tag{3.1}$$

As will be clear from the following (and as can be verified *a posteriori* from the results), we consider the limit in which the presence of the disperse phase does not change the orders of magnitude associated with the wave propagation process in the pure vapour. Hence the phase speed C of the wave can be estimated to be of the order of $(p_0/\rho_0)^{1/2}$, where the subscript 0 denotes equilibrium values, and we take $\lambda\omega = C$. Let

$$M = \frac{\Delta u}{C}, \qquad \Pi = \frac{\Delta p}{p_0}, \tag{3.2}$$

denote the non-dimensional amplitude of the perturbations of the vapour velocity and pressure due to the passage of the wave. From the adiabatic pressure–temperature and pressure–density relations, the latter is also a measure of the non-dimensional temperature and density disturbances and we therefore set

$$\begin{aligned} p &= p_0 \left(1 + \Pi p_* \right), & T &= T_0 \left(1 + \Pi T_* \right), \\ \rho &= \rho_0 \left(1 + \Pi \rho_* \right), & u &= MC u_*. \end{aligned} \tag{3.3}$$
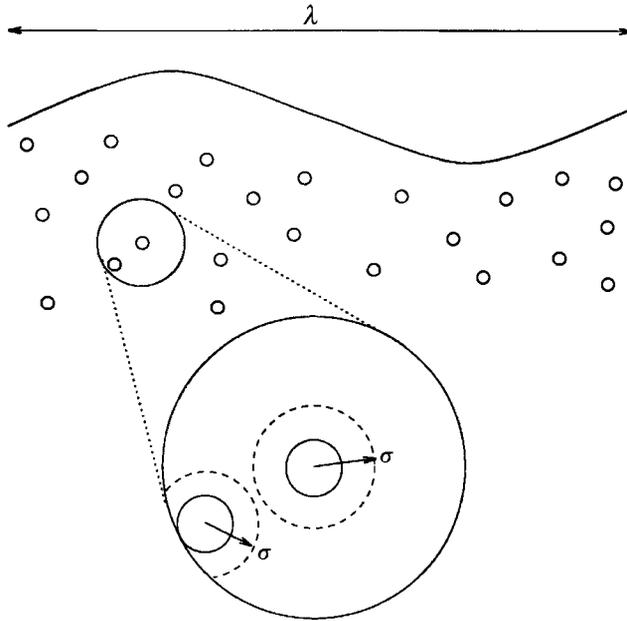
FIGURE 1. Schematic illustration of the present singular perturbation approach. On the macroscopic length scale λ of the wave the droplet size is negligible. Upon 'zooming in' to the local scale the drop radius a becomes finite. The matching of the droplet (inner) and macroscopic (outer) fields is performed at the intermediate scale σ that is the radius of the 'guard sphere' surrounding each drop.

Upon substitution into the vapour momentum equation we find

$$(1 + \Pi \rho_*)\frac{\mathrm{d}u_*}{\mathrm{d}t_*} + \frac{\Pi}{M}\frac{p_0}{\rho_0 C^2}\nabla_* p_* = \epsilon\left[\nabla_*^2 u_* + \eta \nabla_*(\nabla_* \cdot u_*)\right], \tag{3.4}$$

where

$$\epsilon = \frac{\mu}{\rho_0 \lambda C} \tag{3.5}$$

is the inverse of the acoustic Reynolds number associated with the wave. This parameter is very small up to frequencies in the tens of MHz range so that it is the two terms in the left-hand side of (3.4) that must balance, which requires that $\Pi \sim M$. For simplicity, and without loss of generality, we take

$$\Pi = M, \qquad \frac{p_0}{\rho_0 C^2} = 1, \tag{3.6}$$

and we explicitly limit the analysis to weak (linear) waves in the macroscopic sense, $M \ll 1$. As is known from the corresponding problem for bubbly liquids (Caflisch *et al.* 1985), linearization at the macroscopic level does not automatically imply linearization at the microscopic (droplet) level.

With these specifications, the momentum equation (3.4) then becomes

$$\frac{\partial u_*}{\partial t_*} + \nabla_* p_* = \epsilon\left[\nabla_*^2 u_* + \eta \nabla_*(\nabla_* \cdot u_*)\right]. \tag{3.7}$$

After similar considerations the vapour continuity equation becomes

$$\frac{\partial \rho_*}{\partial t_*} + \nabla_* \cdot u_* = 0, \tag{3.8}$$

and the energy equation (2.3)

$$\frac{\partial T_*}{\partial t_*} - \frac{\gamma - 1}{\gamma} \frac{\partial p_*}{\partial t_*} = \frac{\epsilon}{Pr} \nabla_*^2 T_*, \tag{3.9}$$

where γ is the ratio of the specific heats of the vapour and Pr is the Prandtl number, usually of order 1 for gases. The contribution of viscous dissipation is multiplied by $M\epsilon$ and is therefore negligible.

3.2. *Microscopic scaling*

While the characteristic time at the microscopic level is the same as that at the macroscopic level, the characteristic length is the droplet radius a and we set therefore

$$x = a\hat{x}, \qquad \text{or} \qquad x_* = \delta\hat{x}, \tag{3.10}$$

where

$$\delta = \frac{a}{\lambda} \ll 1 \tag{3.11}$$

is the dimensionless drop radius. The velocity scale of the flow in the neighbourhood of a drop is of the same order as in the outer region so that it is appropriate to use the reference velocity Δu for the microscopic region as well. This choice could be inappropriate if the velocity scale Δu_{pc} due to the phase change process were much larger than Δu. The estimate that is provided below in connection with (3.16) proves however that, owing to the magnitude of the latent heat away from the critical point, at most $\Delta u_{pc} \sim \Delta u$. It is because of this circumstance that, unlike the case of a bubbly liquid where the radial velocity of the bubbles can be large even in the presence of a relatively weak wave on the macroscopic scale, here the microscopic momentum equation for the local problem will also be found to be linear when phase change effects are significant.

From the continuity equation (2.1), using (3.6) and neglecting terms of second and higher order in M, we find

$$M \left(\frac{\partial \rho_*}{\partial t_*} + \frac{M}{\delta} u_* \cdot \hat{\nabla} \rho_* \right) + \frac{M}{\delta} \hat{\nabla} \cdot u_* = 0 \tag{3.12}$$

from which, since $\delta \ll 1$,

$$\hat{\nabla} \cdot u_* = 0 \tag{3.13}$$

to leading order.

Before scaling the other equations, let us consider the conditions of conservation of mass and energy at the interface. In the first one, (2.5), we shall assume ρ_0/ρ_L to be small so that the velocity of the interface w can be taken to be essentially equal to u_L. Furthermore, as will be argued shortly, it is a good approximation to neglect the flow in the droplet, so that u_L can be identified with the drop translational velocity v. Upon combining (2.5) and (2.6) we then have

$$(u_* - v_*) \cdot n = -\Lambda \left(K_* \hat{\nabla} T_{L*} + q_* \right) \cdot n, \tag{3.14}$$

where

$$K_* = \frac{k_L}{k}, \qquad q_* = \frac{a}{kMT_0} q. \tag{3.15}$$

The parameter Λ, given by

$$\Lambda = \frac{kT_0}{aL\rho_0 C} = \frac{C_p T_0}{L} \frac{1}{Pr} \frac{\epsilon}{\delta}, \tag{3.16}$$

may be interpreted as the ratio between the phase change velocity Δu_{pc} mentioned before and the particle velocity in the wave Δu. The fraction $C_p T_0/L$ (sometimes called the Jacob number) is typically of order 1 or smaller away from the critical point owing to the large value of the latent heat.

For phase change effects to be significant, Λ must be $O(1)$ or larger, which implies $O(\delta) = O(\epsilon)$ or $O(\delta) < O(\epsilon)$. The Reynolds number Re_D based on the local vapour velocity is

$$Re_D = \frac{2a\rho_0\Delta u}{\mu} = 2M\frac{\delta}{\epsilon} \qquad (3.17)$$

and is therefore small in either case, $O(M)$ or smaller. Viscosity plays then a strong role in the flow in the neighborhood of the drops so that the appropriate scale for the pressure gradient is

$$\nabla p = \mu\frac{\Delta u}{a^2}\hat{\nabla}\hat{p}. \qquad (3.18)$$

By using this scaling the vapour momentum equation (2.2) becomes

$$\left(\frac{\partial}{\partial t_*} + \frac{M}{\delta}u_* \cdot \hat{\nabla}\right)u_* + \frac{\epsilon}{\delta^2}\hat{\nabla}\hat{p} = \frac{\epsilon}{\delta^2}\hat{\nabla}^2 u_*. \qquad (3.19)$$

Since $M \ll 1$, in both cases $O(\delta) = O(\epsilon)$ and $O(\delta) < O(\epsilon)$, the dominant terms give the Stokes equation. We use the relation (3.18) rewritten in the form

$$\hat{\nabla}\hat{p} = \frac{\rho_0 Ca}{\mu}\hat{\nabla}p_*, \qquad (3.20)$$

to write this equation as

$$\hat{\nabla}p_* - \frac{1}{Re}\hat{\nabla}^2 u_* = 0, \qquad (3.21)$$

where the droplet Reynolds number based on the wave velocity is defined by

$$Re = \frac{\delta}{\epsilon} = \frac{\rho_0 Ca}{\mu}. \qquad (3.22)$$

When $O(\delta) = O(\epsilon)$, $Re \sim 1$, which implies that the penetration distance of viscous effects over a wave period, of the order of $(\mu/\rho_0\omega)^{1/2}$, is large compared with the drop radius (more precisely, $a/(\mu/\rho\omega)^{1/2} \sim (\delta)^{1/2}$).

The Knudsen number Kn based on the particle radius can be defined by (Gyarmathy 1982)

$$Kn = \frac{\mu}{\rho_0 aC} = \frac{1}{Re} = \frac{\epsilon}{\delta}, \qquad (3.23)$$

and is therefore of order 1 if $O(\delta) = O(\epsilon)$, while it is large if $O(\delta) < O(\epsilon)$. The latter case corresponds therefore to nearly free-molecular flow and cannot be analysed on the basis of the present equations. It will not be considered further. On the other hand, $Kn \sim 1$ implies that the drop size is comparable to the mean free path so that the prevailing flow regime is transitional between the continuum and the free-molecular limits. As a consequence, a special treatment of the local problem is necessary, as will be seen in §5.

Another distinguished limit of (3.19) is found for $O(\epsilon) = O(\delta^2)$, $O(M) < O(\delta)$. In this case one finds the linearized Navier–Stokes equation

$$\frac{\partial u_*}{\partial t_*} + \frac{\epsilon}{\delta^2}\hat{\nabla}\hat{p} = \frac{\epsilon}{\delta^2}\hat{\nabla}^2 u_*. \qquad (3.24)$$

Now $Kn = O(\delta)$ so that the standard continuum relations apply. However $\varLambda = O(\delta)$ also, so that phase change effects are negligible. The fog behaves therefore no differently than a dusty gas. The interphase momentum interaction acquires the Basset memory term. A third case to consider would be $O(\epsilon) = O(\delta^2)$, $O(M) = O(\delta)$, but it is clear from (3.17) that then $Re_D = O(1)$ rather than small. The condition $O(M) = O(\delta)$ signifies that the particle displacement in the acoustic wave is comparable with the drop radius and, as a consequence, the local problem does not simplify in any way and the full Navier–Stokes equation must be solved. Phase change however remains unimportant.

By using the local scaling, and in particular (3.18) for the pressure term, the energy equation (2.3) becomes

$$\left(\frac{\partial}{\partial t_*} + \frac{M}{\delta} \boldsymbol{u_*} \cdot \hat{\boldsymbol{\nabla}}\right) T_* - \frac{\gamma-1}{\gamma} Re \left(\frac{\partial}{\partial t_*} + \frac{M}{\delta} \boldsymbol{u_*} \cdot \hat{\boldsymbol{\nabla}}\right) \hat{p}$$
$$= \frac{\epsilon}{\delta^2} \left(\frac{1}{Pr} \hat{\nabla}^2 T_* + \frac{\gamma-1}{2\gamma} M \tau_* : \tau_*\right), \qquad (3.25)$$

where

$$\tau_* = \boldsymbol{\nabla_* u_*} + (\boldsymbol{\nabla_* u_*})^T. \qquad (3.26)$$

In the limit $O(\delta) = O(\epsilon)$, $M \ll 1$ that we consider we then find just

$$\hat{\nabla}^2 T_* = 0. \qquad (3.27)$$

This result can be understood in physical terms by noting that, in the conditions of this scaling, the characteristic time for the variation of the vapour enthalpy is the wave period ω^{-1}, which is much longer than the time, $Pr \rho_0 a^2/\mu = Pr \, Re \delta/\omega$, necessary for the propagation of temperature disturbances by conduction. The droplets are therefore surrounded by a quasi-steady temperature environment so that conduction becomes the limiting thermal process.

The local scaling analysis discussed here has focused on the situation where phase change processes are important and has shown that, below the critical temperature where the latent heat is large compared with the sensible heat, this only happens in regimes where the standard continuum theory is not applicable. Conversely, when the latter holds, the coupling between the phases is dominated by momentum – rather than mass – exchange and is therefore uninteresting for the purposes of this work.

The local variables (3.10) are also the appropriate ones for the liquid phase. We shall take the liquid to be quiescent except for the solid-body motion due to the translation of the drops. This approximation amounts to neglecting the gas viscous drag on the liquid at the interface, surface tension variations, inertia, and other smaller effects. The large viscosity of the liquid, the strong effect of surface tension on small drops, and the relatively weak phase change rates furnish a basis for these simplifications. As a consequence of this condition of mechanical quasi-equilibrium, the drops remain very nearly spherical. The only equation to be considered in the liquid is thus the energy equation. Since the droplets are small, and for simplicity, we assume that the steady energy equation is a good approximation in the liquid,

$$\hat{\nabla}^2 T_{L_*} = 0. \qquad (3.28)$$

Unsteady effects could be easily accounted for, if at the price of introducing a memory term in the effective equations.

Because of the linearization of the problem, the saturation condition can be

expressed explicitly from the Clausius–Clapeyron equation in the form

$$p_* = H T_{s*},\tag{3.29}$$

where

$$H = \frac{L}{\mathscr{R}_v T_0}.\tag{3.30}$$

If the Hertz–Knudsen relation (2.10) is used, one finds

$$p_* = H T_{s*} - H' (\mathbf{u}_* - \mathbf{v}_*) \cdot \mathbf{n},\tag{3.31}$$

where T_{s*} is the (dimensionless) surface temperature, H is as before, and H' is given by

$$H' = \frac{(2\pi)^{1/2}}{\alpha}.\tag{3.32}$$

3.3. *Final form of the field equations*

We can now write the final form of the equations to be considered. We obtain this form by using the perfect-gas equation of state to eliminate the density and by including only the terms that are important in the macroscopic or in the microscopic scaling. It is convenient to represent the velocity field in terms of a scalar and a vector potential

$$\mathbf{u}_* = \nabla_*\phi_* + \nabla_* \times A_*,\tag{3.33}$$

with $\nabla_* \cdot A_* = 0$. The vapour energy equation is

$$\frac{\epsilon}{Pr}\nabla_*^2 T_* = \frac{\partial T_*}{\partial t_*} - \frac{\gamma - 1}{\gamma}\frac{\partial p_*}{\partial t_*},\tag{3.34}$$

while, upon elimination of $\partial T_*/\partial t_*$, the continuity equation becomes

$$\nabla_*^2 \left(\phi_* - \frac{\epsilon}{Pr} T_*\right) = -\frac{1}{\gamma}\frac{\partial p_*}{\partial t_*}.\tag{3.35}$$

Upon taking the divergence of the momentum equation we find

$$\nabla_*^2 \left(\frac{\partial \phi_*}{\partial t_*} + p_*\right) = \epsilon \nabla\nabla : \tau_*.\tag{3.36}$$

Finally, by taking the curl of the momentum equation, we have the equation satisfied by the dimensionless vorticity $\omega_* = \nabla_* \times \mathbf{u}_* = -\nabla_*^2 A_*$,

$$\epsilon \nabla_*^2 \omega_* = \frac{\partial \omega_*}{\partial t_*}.\tag{3.37}$$

4. Outline of the method

We now give an outline of the approach that will be followed to obtain the averaged equations and derive a fundamental result that will be used repeatedly in §6. The method will be presented in a heuristic manner. A better quantitative justification is given by Caflisch *et al.* (1985), Caflisch & Rubinstein (1986), and Rubinstein (1986).

We consider a suspension containing N droplets in an otherwise unbounded vapour-filled space. The point of view that we take is that the droplets introduce singular perturbations in a flow field that is otherwise governed by the standard linearized equations for a compressible fluid. The small parameter is δ defined by (3.11) and

the mathematical limit that we consider is $\delta \rightarrow 0$. The matching of the inner field dominated by each drop and the outer field (or, in other words, the overlap domain of the inner and outer fields) occurs at a distance σ from the drop such that

$$\delta \ll \sigma \ll 1. \tag{4.1}$$

It will be shown in §6 that, in order for the droplets to have a non-vanishing effect in the limit $\delta \rightarrow 0$, one must consider simultaneously the limit $N \rightarrow \infty$ in such a way that $N\delta^2$ remains of order 1. The total volume occupied by the droplets, of order $N\delta^3 \sim O(1)\,\delta$, is therefore seen to vanish. In the case studied by Caflisch *et al.* $N\delta$ remained constant in the limit so that the volume fraction was of order δ^2, whereas it is larger, $O(\delta)$, here. That problem is thus more singular than the present one in the sense that fewer inhomogeneities are sufficient to have the same $O(1)$ effect on the flow.

At some given instant we surround each droplet by a sphere of radius σ concentric with it (figure 1). Since the surface of this sphere must be in the matching region, $\delta/\sigma \rightarrow 0$ as $\delta \rightarrow 0$. Furthermore, for reasons that will become clear shortly, we require the total volume occupied by these 'guard spheres' to tend to zero in the limit $\delta \rightarrow 0$ so that

$$N\sigma^3 = N\delta^2 \left(\frac{\sigma}{\delta^{2/3}}\right)^3 \sim O(1) \left(\frac{\sigma}{\delta^{2/3}}\right)^3 \rightarrow 0. \tag{4.2}$$

These conditions will evidently be satisfied if and only if

$$\sigma = O(\delta^p), \qquad \text{with} \quad 2/3 < p < 1, \tag{4.3}$$

which gives a formal definition of the overlap domain.

If the volume of the guard spheres tends to zero in the limit, it is reasonable to assume that the probability of two of them overlapping is negligible. A formal proof of this fact is available for the case studied by Caflisch *et al.* (1985), for which $N \sim 1/\delta$ (see e.g. Caflisch & Rubinstein 1986), but not here. However, this assumption may be made plausible by noting that the mean inter-particle distance ℓ is of the order of the number density to the power $-1/3$, so that $\ell \sim \delta^{2/3}$ from which $\sigma/\ell \rightarrow 0$ by (4.3). This circumstance implies that, in the limit, the particles are well separated on the scale σ.

All the vapour-phase equations of the previous section have a structure of the type

$$\nabla^2 \Psi = s. \tag{4.4}$$

By applying Green's identity to the domain V external to the collection of guard spheres we may write

$$\Psi(x) = \int_V G\,s\,\mathrm{d}^3 x' + \sum_{\alpha=1}^{N} \int_{S^\alpha} (G\nabla'\Psi - \Psi\nabla'G) \cdot n\,\mathrm{d}S'^\alpha, \tag{4.5}$$

where n is the unit normal directed out of the surface S^α of each guard sphere and the time dependence is understood. Here the index α labels the particles and G is the free-space Green's function solution of

$$\nabla^2 G = \delta^{(3)}(x - x'). \tag{4.6}$$

G is symmetric, $G(x - x') = G(x' - x)$, and the spatial scale for its variation in the neighbourhood of either x or x' is $|x - x'|$. For any given configuration of drops, the probability that a point x chosen at random is inside, or close to, a guard sphere is of the order of the volume fraction of the guard spheres and is therefore vanishingly

small in the limit. Hence, except for a set of measure zero, any point x will be 'far' (on the scale of σ) from all the droplets, including the closest ones, as $\delta \to 0$. This remark implies that if, for x' on the guard sphere surrounding the αth droplet, we define a new variable y by

$$x' = x^\alpha + y, \tag{4.7}$$

for all α, and almost all x,

$$|y| = \sigma \ll |x - x^\alpha|. \tag{4.8}$$

In the surface integrals appearing in (4.5) we can therefore use the approximations

$$\left.\begin{aligned}
G(x - x') &= G(x - x^\alpha) - y \cdot \nabla G(x - x^\alpha) + O(\sigma^2), \\
n \cdot \nabla' G(x - x') &= -n \cdot \nabla G(x - x^\alpha) + (y\,n) : \nabla\nabla G(x - x^\alpha) + O(\sigma^2),
\end{aligned}\right\} \tag{4.9}$$

to find

$$\Psi(x) = \int_V G s \, \mathrm{d}^3 x' + \sum_{\alpha=1}^{N} \Big[G I^{(1)}(\Psi, x^\alpha)$$
$$- \nabla G \cdot [I^{(2)}(\Psi, x^\alpha) - I^{(3)}(\Psi, x^\alpha)] - (\nabla\nabla G) : I^{(4)}(\Psi, x^\alpha)\Big]. \tag{4.10}$$

Here the argument of all the G is $x - x^\alpha$ and we have introduced the definitions

$$I^{(1)}(\Psi, x^\alpha) = \int_{S^\alpha} n \cdot \nabla_y \Psi(x^\alpha + y) \, \mathrm{d}S^\alpha, \tag{4.11}$$

$$I^{(2)}(\Psi, x^\alpha) = \int_{S^\alpha} y \, (n \cdot \nabla_y) \Psi(x^\alpha + y) \, \mathrm{d}S^\alpha, \tag{4.12}$$

$$I^{(3)}(\Psi, x^\alpha) = \int_{S^\alpha} n \, \Psi(x^\alpha + y) \, \mathrm{d}S^\alpha, \tag{4.13}$$

$$I^{(4)}(\Psi, x^\alpha) = \int_{S^\alpha} y \, n \Psi(x^\alpha + y) \, \mathrm{d}S^\alpha. \tag{4.14}$$

The key point to note in connection with these integrals is that, since they are taken over a surface in the inner–outer matching region, they can be calculated by using the inner solution for Ψ. It is this circumstance that enables us to obtain explicit expressions. It will be seen below that, for the cases we consider, the local fields are the sum of a constant and a part decaying as some power of the distance from the drops' centres. With this structure, $I^{(2)}$ and $I^{(3)}$ have the same order of magnitude, while $I^{(1)}$ is $O(1/\sigma)$ larger. This will therefore be the dominant term, unless it happens to vanish. The last integral $I^{(4)}$ is instead $O(\sigma)$ smaller, and therefore never significant. Accordingly, we shall drop it henceforth. Further terms in the expansions (4.9) would give contributions of still higher order in σ and are therefore negligible.

Following Caflisch *et al.* (1985), we now pass to the continuum limit. We consider $N \to \infty$ in such a way that a continuous droplet number density per unit volume $n(x)$ can be defined in the limit with the property

$$\sum_{\alpha=1}^{N} \to \int_V \mathrm{d}^3 x' \, n(x'). \tag{4.15}$$

Denoting by an overbar the averaged fields resulting from this operation, we then

re-write (4.10) as

$$\overline{\Psi}(x) = \int_V G\bar{s}\,d^3x' + \int_V d^3x'\,n(x')\left\{G\,I^{(1)}(\Psi,x') - \nabla G\cdot[I^{(2)}(\Psi,x') - I^{(3)}(\Psi,x')]\right\}.$$
(4.16)

Since, in our limit, the drop volume fraction tends to zero, the field $\overline{\Psi}$ may be considered to be defined over the whole space. The substitution of $\bar{s}$ for s in the volume integral presupposes that the latter quantity is smoothly varying. More precisely, as in the specific cases that occur below, suppose that in the neighbourhood of each droplet

$$s - \bar{s} \sim O(\delta^i r^{-j}).$$
(4.17)

If we break up the volume integral into a part exterior and a part interior to the guard spheres, the contribution to the final result arising from the 'near-field' behaviour in the latter is then of the order of $N\delta^i\sigma^{3-j}$. As mentioned before, in the continuum limit, $N = O(\delta^{-2})$. Hence, by (4.3), we deduce that the substitution of $\bar{s}$ for s is valid as $\delta \to 0$ provided that

$$i - 2 + p(3 - j) > 0,$$
(4.18)

with p as in (4.3). It will be necessary to check this condition for each one of our equations.

Since the droplets are only present in a finite volume, n is zero at infinity and one can carry out an integration by parts to find

$$\overline{\Psi}(x) = \int_V G\bar{s}\,d^3x' + \int_V d^3x'\,G(x - x')\left\{n(x')\,I^{(1)}(\Psi,x')\right.$$
$$\left. + \nabla\cdot[n(x')\left(I^{(2)}(\Psi,x') - I^{(3)}(\Psi,x')\right)]\right\}.$$
(4.19)

At this point application of the Laplacian ∇^2 and use of (4.6) yields the averaged equation satisfied by $\overline{\Psi}$,

$$\nabla^2\overline{\Psi} = \bar{s} + n(x)\,I^{(1)}(\Psi,x) + \nabla\cdot\{n(x)[I^{(2)}(\Psi,x) - I^{(3)}(\Psi,x)]\}.$$
(4.20)

This result will be used repeatedly in §6. Note that the dimensionless droplet number density n appearing here equals the dimensional one divided by λ^3.

The case of particles with a distribution of radii (but still such that the present scaling applies) may be treated by a slight modification of the continuum limit rule (4.15), namely

$$\sum_{\alpha=1}^{N} \to \int_V d^3x' \int da\,n(x',a),$$
(4.21)

where $n(x',a)$ is now the particle number density per unit radius increment and the integration over a is carried out over the range of particle radii. With this generalization, the averaged equation (4.20) takes the form

$$\nabla^2\overline{\Psi} = \bar{s} + \int da\,n(x,a)\,I^{(1)}(\Psi,x,a) + \nabla\cdot\int da\,n(x,a)[I^{(2)}(\Psi,x,a) - I^{(3)}(\Psi,x,a)].$$
(4.22)

For simplicity, we will however limit ourselves to the monodisperse case (4.20) in what follows. Rubinstein (1986) presents a formulation capable of dealing with a wide distribution of drop radii.

5. The local problem

We now obtain a solution of the local problem that will enable us to calculate explicit expressions for the integrals $I^{(\ell)}$ appearing in the averaged equations (4.20). The equations to be solved are (3.13), (3.21), and (3.27) subject to the boundary conditions of matching with the averaged fields at large distances from the generic drop considered. It is this condition – as opposed to matching with the 'bare' incident fields – that characterizes the approach of Foldy (1945) to this type of problems. This step introduces a coupling with the other inhomogeneities.

As usual with matching, the relevant values of the outer fields are those obtained in the inner limit. In the present problem these coincide with the values assumed at the centre x^α of the generic droplet. In this section all quantities are made dimensionless according to the local scaling. However, we omit any specific indication of this fact for simplicity of writing.

A difficulty that arises is that we are concerned with a transitional situation between continuum and free-molecular flow near the drops so that some of the usual continuum relations are not applicable. However, it will be found in the next section that the local solution influences the effective equations chiefly through the mean drag and evaporation rate of the drops for which empirical information is available. Additional properties of the local solution are only required to estimate the order of magnitude of terms that are neglected. Since the local flow is characterized by a Knudsen number of order unity, rather than large, one does not expect order-of-magnitude discrepancies between the real local fields and those obtained from the continuum equations. Hence, the latter can safely be used for estimation purposes. This approach can also be justified by recalling a standard approximation in the finite-Kn-regime (see e.g. Gyarmathy 1982), in which the flow field is divided into two zones. Molecular effects are accounted for up to a distance of about one mean free path from the drop, while the usual continuum relations are used further away. It will be recalled that, in the present application, the inner domain extends from the drop all the way to a distance of order $\sigma \gg \delta$. We then deduce that the usual continuum laws are approximately valid over most of the inner domain and, in particular, in the matching region.

In order to find the microscopic vapour velocity field it is expedient to use the drop's rest frame and to set, as in (3.33),

$$u' \equiv u - v = \nabla\phi' + \nabla \times A',\tag{5.1}$$

where v is the drop velocity. At large distance from the drop, u must match the outer field $\bar{u}(x^\alpha, t)$ and therefore

$$u' \rightarrow \bar{u}(x^\alpha, t) - v \equiv U.\tag{5.2}$$

It may be recalled that, according to the local scaling analysis of §3, time does not enter explicitly in the local momentum equation and this variable should therefore be considered as a parameter. A suitable choice for ϕ' is

$$\phi' = U \cdot r - \frac{\dot{M}}{4\pi r},\tag{5.3}$$

where $r \equiv x - x^\alpha$, $r = |r|$, and $\dot{M}$ is the drop evaporation rate. It will be observed that $\nabla^2\phi' = 0$ as required by (3.13).

Following the argument given in Landau & Lifshitz (1959, section 20) we note that A' is an axial vector that can only depend on r, U, and $\nabla\bar{T}(x^\alpha, t)$. The last quantity is introduced because the flow field may in general depend on nonuniformities in

the evaporation rate over the drop's surface, which can be induced by temperature gradients in the far-field temperatures. The reason not to include higher-order derivatives of $\overline{T}$ is that, by dimensional reasoning, such terms would be smaller by some power of $a/\lambda = \delta$ and therefore negligible to the present order. Since A' must depend linearly on U and $\nabla\overline{T}$, as in Landau & Lifshitz we are led to $A' = \nabla \times \left[F(r)U + G(r)\nabla\overline{T}\right]$. However, U and $\nabla\overline{T}$ are due to the same acoustic wave and therefore, at least locally near the drop, they must be parallel so that, without loss of generality, we may write

$$A' = \nabla \times [f(r)U] \,, \tag{5.4}$$

just as in the non-evaporating case. The reasoning of Landau & Lifshitz can then be followed without further modification and leads to

$$f = \frac{3r}{4A} + \frac{B}{r} \,, \tag{5.5}$$

where A and B are constants, from which

$$u = \overline{u} - \frac{3}{4Ar}\left[(U \cdot e_r)e_r + U\right] + \frac{B}{r^3}\left[3(U \cdot e_r)e_r - U\right] + \frac{\dot{M}}{4\pi r^2}e_r \,, \tag{5.6}$$

where $e_r = r/r$. The associated pressure field is

$$p = \overline{p} - \frac{3}{2A\,Re\,r^2}U \cdot e_r \,, \tag{5.7}$$

and the vorticity

$$\omega = \frac{3}{2Ar^2}e_r \times U \,. \tag{5.8}$$

The constant A is directly related to the (dimensionless) drag force, F_D, on the drop (Landau & Lifshitz 1959),

$$F_D = \frac{6\pi}{A}U \,. \tag{5.9}$$

When continuum theory applies, the no-slip condition is satisfied, and there is no phase change, $A = 1$ and (5.9) is just the Stokes drag law. Several expressions have been proposed to correct this result when the Knudsen number is not negligible (see e.g. Millikan 1923; Schaaf & Chambré 1961; Gyarmathy 1982; Cercignani 1988). Gyarmathy favours the form

$$C_D = \frac{C_D^c}{1 + (C_D^c Re_D/8B_F)Kn} \,, \tag{5.10}$$

where C_D is the drag coefficient, equal to $24/Re_D$ in ordinary Stokes flow, the superscript c denotes the continuum-theory value, and $B_F \approx 1.48$. Upon using (5.9) in this relation, one finds

$$\frac{1}{A} = \frac{1/A^c}{1 + 2.02(Kn/A^c)} \,. \tag{5.11}$$

We cannot set $A^c = 1$ here because, as will be shown shortly, phase change affects the value of A even in the absence of slip.

When either saturation or the Hertz–Knudsen relation (2.10) is imposed, it is evident from (5.6) and (5.7) that only an isotropic term and a term proportional to $U \cdot e_r$ can appear in the expression for the vapour temperature at the drop surface

and we therefore write

$$T|_{r=1} = \overline{T} + \Theta_0 + \Theta_1 U \cdot e_r . \tag{5.12}$$

According to the liquid energy equation (3.28), the temperature field in the liquid is harmonic and, since it must reduce to (5.12) at the surface according to (2.9), it must have the form

$$T_L = \overline{T} + \Theta_0 + \Theta_1 U \cdot r . \tag{5.13}$$

A corresponding deduction for the vapour temperature is not legitimate because Fourier's law breaks down near the drop and the energy equation in the form (3.27) is not applicable. Empirical expressions for the heat flux Q_0 averaged over the surface are however available, typically in the form

$$Q_0 = \frac{4\pi}{q(Kn)}(\tilde{T} - \overline{T}), \tag{5.14}$$

where $\tilde{T}$ is the surface-averaged temperature and $q(Kn) \to 1$ for $Kn \to 0$. From an expression similar to (5.10) given by Gyarmathy (1982, equation 123) we find, for $\gamma = 1.4$,

$$q = 1 + 1.462\frac{Kn}{Pr} . \tag{5.15}$$

Since, as is clear from (5.12), $\tilde{T} - \overline{T} = \Theta_0$, the constant Θ_0 can be expressed in terms of Q_0 from (5.14):

$$Q_0 = \frac{4\pi}{q} \Theta_0 . \tag{5.16}$$

From (5.6), (5.12), and the Hertz–Knudsen relation (2.10) we readily find

$$H\Theta_0 = \overline{p} - H\overline{T} + \frac{H'}{4\pi} \dot{M} , \qquad H\Theta_1 = H'\left(1 - \frac{3}{2A} + 2B\right) - \frac{3}{2A\,Re} . \tag{5.17}$$

We write the heat flux at the drop's surface in the following form suggested by the need to satisfy the conservation of energy condition (3.14) with $\overline{T}_L$ given by (5.13):

$$q \cdot n = \frac{1}{4\pi}Q_0 + Q_1 U \cdot e_r . \tag{5.18}$$

Then, from the relation (3.14) between the evaporative flux and the energy balance at the interface, we find

$$-AQ_0 = \dot{M} , \qquad -A(Q_1 + K\Theta_1) = 1 - \frac{3}{2A} + 2B . \tag{5.19}$$

The dimensionless K appearing here is defined in (3.15). On the basis of the Clausius–Clapeyron relation (3.29) one can define a saturation temperature by

$$\overline{T}_{sat} = \frac{\overline{p}}{H} . \tag{5.20}$$

From the previous relations one then readily finds

$$Q_0 = -\frac{4\pi}{q + H'/HA}\left(\overline{T} - \overline{T}_{sat}\right) , \qquad \dot{M} = \frac{4\pi A}{q + H'/HA}\left(\overline{T} - \overline{T}_{sat}\right) . \tag{5.21}$$

The previous results do not constitute a complete solution of the problem but are sufficient for the present purposes. To go beyond them one would require a further relation between Q_1 and Θ_1, with the aid of which the constant B would be

determined from (5.17), (5.19). In principle this step requires an analysis of the local problem on the basis of kinetic theory. Recent progress in this direction is a paper by Sone, Takata & Wakabayashi (1994), who however assumed a uniform temperature over the surface of the drop.

In order to calculate the continuum-theory value A^c appearing in (5.11), and for the purpose of demonstrating the nature of a full solution to the problem, we now complete the solution assuming the validity of Fourier's law and no tangential slip at the drop surface (see also Duraiswami 1990). From the former condition we readily find

$$Q_1 = 2\Theta_1, \tag{5.22}$$

while the latter one gives

$$\frac{3}{4A^c} + B = 1. \tag{5.23}$$

Upon using these relations to express Θ_1 and B in the second of (5.17) and of (5.19) one finds

$$-\frac{6}{Q_1} = 2\,Re\,H + A(K+2)(1+2\,Re\,H'), \qquad \frac{1}{A^c} = \frac{2\,Re\,[H + H'A(K+2)]}{2\,Re\,H + A(K+2)(1+2\,Re\,H')}. \tag{5.24}$$

When the saturation condition at the drop surface is applicable, $H' = 0$ and the latter result becomes

$$A^c = 1 + \frac{A}{2\,Re\,H}\,(K+2). \tag{5.25}$$

In dimensional variables this relation can be expressed in the form

$$A^c = 1 + \left(\frac{h_1}{a}\right)^2, \tag{5.26}$$

where the characteristic length h_1 only depends on the physical properties of the fluid and is given by

$$h_1 = \frac{\mu}{\rho_0 L}\left[\frac{C_p T_0}{2Pr}\left(\frac{k_L}{k}+2\right)\right]^{1/2}. \tag{5.27}$$

A graph of h_1 in μm versus temperature for saturated water and water vapour is shown in figure 2 to give an idea of the magnitude of the predicted deviation from the standard Stokes drag law. Depending on the drop radius, the correction can be significant, particularly at lower temperatures where the density is low and the evaporation velocities therefore larger (note the strong sensitivity of the parameter A to the vapour density). An alternative way to examine this point is to rewrite (5.25) explicitly bringing out its dependence on the Knudsen number,

$$A^c = 1 + Kn^2\left[\frac{\gamma-1}{2Pr}\left(\frac{C_p T_0}{L}\right)^2\left(\frac{k_L}{k}+2\right)\right]. \tag{5.28}$$

The quantity in square brackets is also plotted as a function of temperature in the case of water in figure 3. Again we conclude that, when $Kn \sim 1$, the effect of phase change on the drag may not be negligible.
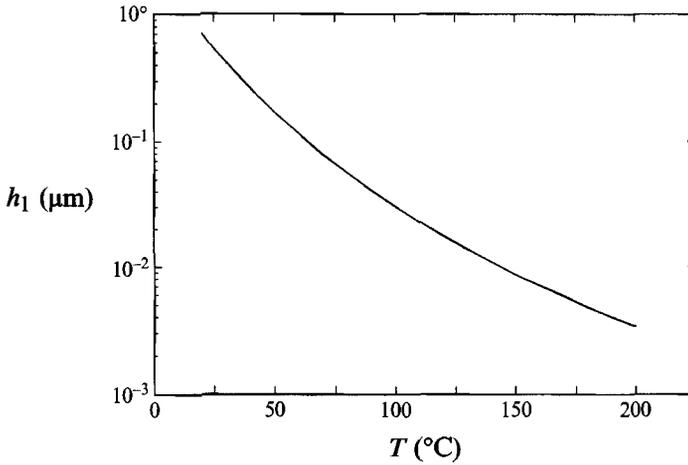
FIGURE 2. Graph of the characteristic length h_1 defined in (5.27) *vs.* temperature for a water-vapour fog.
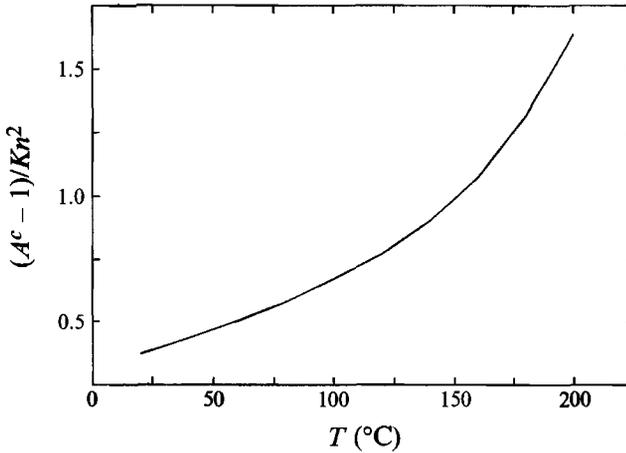
FIGURE 3. Graph of the quantity in square brackets in (5.28) *vs.* temperature for a water-vapour fog.

6. Averaged equations

We can now obtain averaged equations by substituting the local results of the preceding section into the integrals (4.14) that arise in the application of the procedure outlined in section 4. In so doing a clearer statement of the domain of validity of the final result will be derived and several aspects of the technique will be elucidated. We give a detailed derivation for the case of the equation of continuity. The procedure is similar for the other effective equations. All quantities are made dimensionless according to the macroscopic scaling of section 3.1, but asterisks are omitted for simplicity of writing.

6.1. *Continuity*

We start from the continuity equation (3.35). With the notation of §4 we thus have $s = -(1/\gamma)\partial p/\partial t$ and $\Psi = \phi - (\epsilon/Pr)T$.

In outer variables the results (5.3) and (5.7) of the previous section are, after

suppression of the asterisks,

$$p = \bar{p} - \frac{3\delta^2}{2A\,Re\,r^3}(\bar{u} - v)\cdot r,\qquad(6.1)$$

$$\phi = \bar{u}\cdot r - \frac{\delta^2}{4\pi r}\dot{M}.\qquad(6.2)$$

It should be recalled that here $\bar{u}$ is evaluated at the centre of the particle and is therefore independent of r. From (6.2) we then find

$$n\cdot\nabla_y\phi(x^\alpha + y)|_{S^\alpha} \equiv \left.\frac{\partial\phi}{\partial r}\right|_{r=\sigma} = \bar{u}\cdot e_r + \frac{\delta^2}{4\pi\sigma^2}\dot{M}.\qquad(6.3)$$

The integral $I^{(1)}$ is therefore

$$\int_{S^\alpha} n\cdot\nabla_y\phi(x^\alpha + y)\,dS^\alpha = \delta^2\dot{M},\qquad(6.4)$$

since the first term of (6.3) gives a vanishing contribution. Here $\dot{M}$ is given by (5.21). Proceeding in a similar fashion we find for $I^{(3)}$

$$\int_{S^\alpha} n\,\phi\,dS^\alpha = \tfrac{4}{3}\pi\sigma^3\bar{u},\qquad(6.5)$$

with an identical result for $I^{(2)}$. The two contributions $I^{(2)}$ and $I^{(3)}$ happen therefore to balance in this case. Were this accidental cancellation not to occur, one would need to require that this contribution be smaller than the $O(\delta^2)$ one of $I^{(1)}$ for the result to be independent of the matching length scale σ. Again then we would find for σ the order estimate (4.3) of §4.

The integral $I^{(1)}$ for the temperature field is

$$\int_{S^\alpha} n\cdot\nabla_y T\,dS^\alpha.\qquad(6.6)$$

This integral is proportional to the total heat crossing the guard sphere. By conservation of energy, this must equal the total heat exchanged by the droplet with its surroundings and therefore we conclude that

$$\int_{S^\alpha} n\cdot\nabla_y T\,dS^\alpha = -Q_0\,\delta.\qquad(6.7)$$

In order to show that the other integrals give negligible contributions, we note that (3.27) is applicable at some distance from the drop so that, in the matching region, the inner temperature field must have the form

$$T = \bar{T} + \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Theta'_{km}\left(\frac{\delta}{r}\right)^{k+1}Y_k^m,\qquad(6.8)$$

where the Θ'_{km} are constants and the Y_k^m spherical harmonics. With this expression, for the integral $I^{(3)}$ we find

$$\int_{S^\alpha} n\,T\,dS^\alpha = \delta^2\sum_{m=-1}^{1}\Theta_{1m}\int n Y_1^m d\Omega,\qquad(6.9)$$

where we have set $dS^\alpha = \sigma^2 d\Omega$ with Ω the solid angle. A similar expression is found

for $I^{(2)}$. Since these integrals need to be multiplied by ϵ (because for this calculation $\Psi = \phi - (\epsilon/Pr)T$), their contribution vanishes as $\delta \to 0$.

The final step is to check the order of the source term $\partial p/\partial t$. From (6.1) we have

$$\frac{\partial p}{\partial t} - \frac{\partial \overline{p}}{\partial t} \sim O\left(\frac{\delta^2}{r^2 Re}\right), \tag{6.10}$$

where Re is $O(1)$. With the previous estimate (4.3) we see that the condition (4.18) is satisfied so that the source term can be averaged directly.

Upon insertion of these results into the general form (4.10) we then find

$$\overline{\phi} - \frac{\epsilon}{Pr}\overline{T} = -\frac{1}{\gamma}\int G\frac{\partial \overline{p}}{\partial t}\,\mathrm{d}^3x' + \delta^2 \sum_{\alpha=1}^{N} G\left(\dot{M} + \frac{\epsilon}{\delta Pr}Q_0\right). \tag{6.11}$$

The second term in the left-hand side is negligible as $\epsilon \to 0$, but the fraction ϵ/δ in the right-hand side is $O(1)$ in the limit we consider. It is therefore clear from this result that, for consistency of the procedure employed in its derivation, an upper limit on the drop number is

$$N\delta^2 = O(1), \tag{6.12}$$

as mentioned before in §4.

Having established this relation, we can now use the general result (4.20) to cast the previous equation in differential form to find, by the first of (5.19) relating Q_0 and $\dot{M}$,

$$\nabla \cdot \overline{u} + \frac{1}{\gamma}\frac{\partial \overline{p}}{\partial t} = \delta^2 n\left(1 - \frac{\epsilon}{\delta Pr \Lambda}\right)\dot{M}. \tag{6.13}$$

In this and the previous equations $\dot{M}$ should be viewed as the average evaporation rate per drop. An expression for this quantity is given by (5.21).

6.2. *Energy*

The calculation is very similar to that for the energy equation (3.34). The estimate of the source terms proceeds as before, and the result (6.7) can be used directly to find

$$\frac{\partial \overline{T}}{\partial t} - \frac{\gamma - 1}{\gamma}\frac{\partial \overline{p}}{\partial t} = -\frac{n\delta\epsilon}{\Lambda Pr}\dot{M}. \tag{6.14}$$

It may be noted that the validity of this conclusion depends on the assumption that the difference between $\partial T/\partial t$ and $\partial \overline{T}/\partial t$ gives a negligible contribution to the volume integral. This can readily be checked for T given by (6.8), which begins to be applicable a few mean free paths away from the drop surface. While we do not have an expression for the temperature near the surface, we do not expect this region to invalidate this estimate. In the first place, as noted before, the (dimensionless) mean free path is $O(\delta)$ and, since $\delta \ll \sigma$, most of the contribution to the integral comes from the region where (6.8) holds. Secondly, the difference between (6.8) and the actual temperature field is probably small anyway (see e.g. Landau & Lifshitz 1981, section 14).

Upon using the (linearized) equation of state of the vapour and (6.14) in the right-hand side of (6.13), the averaged continuity equation can be cast in the expected form

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot \overline{u} = n\delta^2\dot{M}, \tag{6.15}$$

with $\dot{M}$ from (5.21).

6.3. The disperse-phase equations

Before considering the averaged momentum equation we need to introduce the average velocity of the disperse phase. With the expression (5.9) for the drag force, the (dimensionless) equation of motion of the generic drop α is given by

$$\frac{\partial v^\alpha}{\partial t} = \frac{9}{2 A Re} \left(\frac{\rho_0/\rho_L}{\delta} \right) (\bar{u} - v^\alpha). \tag{6.16}$$

In order to retain as much physics as possible in this relation we assume that, formally, $O(\rho_0/\rho_L) = O(\delta)$, so that the fraction in parentheses is of order 1. It may be recalled that ρ_0/ρ_L had already been assumed small in order to approximate the interface velocity by the drop velocity just before (3.14).

Following Caflisch et al. (1985), as $N \to \infty$, we define the continuum limit $\bar{v}(x, t)$ of the quantity v^α defined at the N positions x^α occupied by the droplets by

$$\frac{1}{N} \sum_{\alpha=1}^{N} [v^\alpha - \bar{v}(x, t)]^2 \to 0. \tag{6.17}$$

On this basis, from (6.16), we may write

$$\frac{\partial \bar{v}}{\partial t} = \frac{9}{2 A Re} \left(\frac{\rho_0/\rho_L}{\delta} \right) (\bar{u} - \bar{v}). \tag{6.18}$$

Although not necessary for the closure of the system, we may also give an equation for the average temperature field in the droplet phase. To this end we note that, as is immediate from (5.13), the mean droplet temperature equals $\bar{T} + \Theta_0$. Upon differentiation with respect to time and use of the relation (5.16) between Θ_0 and Q_0 we then have

$$\frac{\partial \bar{T}_L}{\partial t} = \frac{\partial \bar{T}}{\partial t} - \frac{q}{4\pi A} \frac{\partial \dot{M}}{\partial t}. \tag{6.19}$$

Note that the local problem depends on time parametrically through the matching conditions, although not dynamically through its governing differential equations.

6.4. Momentum

For the momentum balance we have the two equations (3.36) and (3.37). For the first one we cannot apply directly (4.20) because the right-hand side does not give a negligible contribution in the neighbourhood of the drops. Hence we start from the Green's function form, (4.5), which, in this case, is

$$\frac{\partial \phi}{\partial t} + p = \epsilon \int_V G\nabla\nabla : \tau \, d^3x' + \sum_{\alpha=1}^{N} \int_{S^\alpha} \left(\frac{\partial \phi}{\partial t} \nabla G - G \nabla \frac{\partial \phi}{\partial t} \right) \cdot n \, dS^\alpha$$

$$+ \sum_{\alpha=1}^{N} \int_{S^\alpha} (p\nabla G - G\nabla p) \cdot n \, dS^\alpha. \tag{6.20}$$

Upon a twofold integration by parts of the first term we find

$$\frac{\partial \phi}{\partial t} + p = \epsilon \int_V \tau : \nabla\nabla G \, d^3x' + \sum_{\alpha=1}^{N} \int_{S^\alpha} \left(\frac{\partial \phi}{\partial t} \nabla G - G \nabla \frac{\partial \phi}{\partial t} \right) \cdot n \, dS^\alpha$$

$$+ \sum_{\alpha=1}^{N} \int_{S^\alpha} \nabla G \cdot (pn - \epsilon\tau \cdot n) \, dS^\alpha + \sum_{\alpha=1}^{N} \int_{S^\alpha} G n \cdot (\nabla p - \epsilon\nabla \cdot \tau) \, dS^\alpha. \tag{6.21}$$

In terms of τ the momentum equation (3.21) is just

$$\nabla p - \epsilon \nabla \cdot \tau = 0, \tag{6.22}$$

which shows that the terms in the last summation all vanish. (Recall that, since the surfaces S^α are in the matching region far from the drop, the ordinary continuum equations are applicable there.) The term ∇G can be removed from the integrands in the next to the last summation by a Taylor series expansion as in (4.9). To leading order, the remaining term is just the total vapour force on the spherical surface of radius σ. Since there are no momentum sources inside this sphere, and since inertial effects are negligible in the inner region, this force must equal the drag on the enclosed particle given in dimensionless form by (5.9). Hence

$$F^\alpha = \int_{S^\alpha} (p\boldsymbol{n} - \epsilon \tau \cdot \boldsymbol{n}) \, \mathrm{d}S^\alpha = \frac{6\pi}{A} \delta \epsilon \, (\bar{\boldsymbol{u}} - \boldsymbol{v}), \tag{6.23}$$

so that

$$\frac{\partial \phi}{\partial t} + p = \epsilon \int_V \tau : \nabla\nabla G \, \mathrm{d}^3 x' + \sum_{\alpha=1}^N \int_{S^\alpha} \left(\frac{\partial \phi}{\partial t} \nabla G - G \nabla \frac{\partial \phi}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}S^\alpha + \sum_{\alpha=1}^N F^\alpha \cdot \nabla G(\boldsymbol{x} - \boldsymbol{x}^\alpha). \tag{6.24}$$

At this point the procedure leading from (4.5) to (4.20) can be followed to find, as $\epsilon \to 0$,

$$\nabla^2 \left(\frac{\partial \bar{\phi}}{\partial t} + \bar{p} \right) = n\delta^2 \frac{\partial \dot{M}}{\partial t} - \nabla \cdot \left[\frac{6\pi}{A} n\delta \epsilon \, (\bar{\boldsymbol{u}} - \bar{\boldsymbol{v}}) \right], \tag{6.25}$$

with error terms as before.

Lastly we consider the vorticity equation (3.37). In outer coordinates the result (5.8) for the local vorticity is

$$\omega = \frac{3\delta^2}{2A r^3} \, \boldsymbol{r} \times (\bar{\boldsymbol{u}} - \boldsymbol{v}). \tag{6.26}$$

The source term in this case is $s = (1/\epsilon)\partial\omega/\partial t$ and, since $\epsilon = O(\delta)$, the criterion (4.18) is not satisfied and this term cannot be averaged directly. However

$$\int \frac{\partial \omega_i}{\partial t} G \, \mathrm{d}^3 x' \equiv \frac{\partial}{\partial t} \int \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} G \, \mathrm{d}^3 x'$$

$$= \frac{\partial}{\partial t} \int \epsilon_{ijk} \frac{\partial G}{\partial x_j} u_k \, \mathrm{d}^3 x' + \sum_{\alpha=1}^N \int_{S^\alpha} \epsilon_{ijk} n_j \frac{\partial u_k}{\partial t} G \, \mathrm{d}S^\alpha, \tag{6.27}$$

and the inner momentum equations show the last term to vanish. The source term in the first integral now satisfies the criterion (4.18) and can be averaged.

Upon using the explicit expression (6.26), it is found that the integral $I^{(1)}$ vanishes while

$$I^{(2)} - I^{(3)} = \int_{S^\alpha} \left[\boldsymbol{y} \, (\boldsymbol{n} \cdot \nabla_y) \, \omega - \boldsymbol{n} \, \omega \right] \, \mathrm{d}S^\alpha = \frac{6\pi}{A} \delta \epsilon \, (\boldsymbol{v} - \bar{\boldsymbol{u}}), \tag{6.28}$$

with error terms as before. Collecting these contributions the final result for the averaged vorticity equation is then, to leading order,

$$\frac{\partial \bar{\omega}}{\partial t} = -\nabla \times \left[\frac{6\pi}{A} n\delta \epsilon (\bar{\boldsymbol{u}} - \bar{\boldsymbol{v}}) \right]. \tag{6.29}$$

The two equations obtained here specify the divergence and the curl of the velocity

field $\bar{u}$. It is not difficult to prove (Duraiswami 1990) with the help of suitable gauge transformations and the condition of vanishing at infinity (which has already been extensively used, e.g. in writing Green's identity) that they are equivalent to

$$\frac{\partial \bar{u}}{\partial t} + \nabla \bar{p} = -\frac{6\pi}{A} n \delta \epsilon (\bar{u} - \bar{v}), \tag{6.30}$$

as expected. An expression for the drag correction A is given in (5.11) with A^c given in (5.24).

7. The problem for multi-component fogs

We now consider the effects of diffusion on the average equations derived in the previous section. The differences with the one-component case are minor and an abbreviated treatment will be sufficient.

The total density of the gaseous mixture is

$$\rho = \rho_g + \rho_v, \tag{7.1}$$

where the subscripts g and v refer to the incondensible gas and the vapour respectively. The local composition of the gaseous mixture is specified in terms of the vapour mass concentration c defined by

$$c = \frac{\rho_v}{\rho}. \tag{7.2}$$

We shall neglect the solubility of the incondensible gas in the liquid phase so that the entire mass flux crossing the interface from the vapour side is due to the vapour component alone.

With

$$c = c_0 (1 + \Pi c_*) \tag{7.3}$$

(cf. (3.3)), the dimensionless linearized equation of state of the mixture is

$$\rho_* = p_* - T_* - Wc_*, \tag{7.4}$$

where

$$W = \frac{c_0(\mathscr{R}_v - \mathscr{R}_g)}{\mathscr{R}_v c_0 + \mathscr{R}_g(1 - c_0)}, \tag{7.5}$$

with $\mathscr{R}_g$ the specific gas constant (cf. (2.8)).

As mentioned in §2, the continuity and momentum equations remain unchanged. For the linearized dimensionless energy equation we find

$$\frac{\partial T_*}{\partial t_*} - \frac{\hat{\gamma} - 1}{\hat{\gamma}} \frac{\partial p_*}{\partial t_*} = \frac{\epsilon}{Pr} \nabla_*^2 T_*, \tag{7.6}$$

which is formally identical to (3.9) with the replacement of γ by $\hat{\gamma}$ defined by

$$\hat{\gamma} = \frac{c_0 C_{p_v} + (1 - c_0) C_{p_g}}{c_0 C_{v_v} + (1 - c_0) C_{v_g}}, \tag{7.7}$$

where the C_v are the specific heats at constant volume. The linearized diffusion equation in outer variables is

$$\frac{\partial c_*}{\partial t_*} = \frac{\epsilon}{Sc} \nabla_*^2 c_*, \tag{7.8}$$

where $Sc = \mu/\rho_0 \mathscr{D}$ is the Schmidt number. Upon expressing the velocity in terms of

a scalar and vector potential as in (3.33), use of the equation of state (7.4), and of the heat and mass diffusion equations (7.6) and (7.8), the equation of continuity may be written

$$\nabla_*^2 \left(\phi_* - \frac{\epsilon}{Pr} T_* - \frac{\epsilon}{Sc} c_* \right) = -\frac{1}{\hat{\gamma}} \frac{\partial p_*}{\partial t_*}, \qquad (7.9)$$

which is very similar to the single-component case (3.35).

Upon performing a local scaling analysis parallel to that carried out in §3, since the Schmidt number is of order one for gases, one finds the same limits and analogous equations for the conservation of mass, momentum, and energy. For the diffusion equation, in the scaling $\epsilon \sim \delta$, the result is

$$\hat{\nabla}^2 c_* = 0, \qquad (7.10)$$

although, as before in the case of (3.27), the presence of non-negligible kinetic effects necessitates some correction in the neighbourhood of the drop.

At the interface we apply the equation of conservation of the total mass (2.5) and of the total energy (2.6) as before. With the neglect of the solubility of the incondensible gas in the liquid forming the drop, the total mass flux of the incondensible gas at the surface of the drops must vanish from which we have

$$(\boldsymbol{u}_* - \boldsymbol{v}_*) \cdot \boldsymbol{n} = \Gamma \, \boldsymbol{n} \cdot \boldsymbol{j}_*, \qquad (7.11)$$

where

$$\Gamma = \frac{\epsilon}{\delta Sc} \frac{c_0}{1 - c_0}, \qquad (7.12)$$

and

$$\boldsymbol{j}_* = \frac{1}{\Pi} \frac{a}{\rho_0 c_0 \mathscr{D}} \boldsymbol{j}, \qquad (7.13)$$

is the dimensionless diffusion mass flux. When Fick's law applies we have

$$\boldsymbol{j}_* = -\hat{\nabla} c_*. \qquad (7.14)$$

Analogously to the parameter Λ, Γ also is negligibly small unless $O(\delta) = O(\epsilon)$ or $O(\delta) < O(\epsilon)$, i.e. except when the droplets are of the order of the mean free path or smaller.

The saturation relation corresponding to (3.29) is now

$$p_* = H T_* - \mathscr{A} c_*, \qquad (7.15)$$

with H as given before in (3.30) and

$$\mathscr{A} = \frac{\mathscr{R}_g}{\mathscr{R}_g + (\mathscr{R}_v - \mathscr{R}_g) c_0} = \frac{\mathscr{R}_g \rho_0 T_0}{p_0}. \qquad (7.16)$$

The analogue of the Hertz–Knudsen relation (2.10) is

$$p_* = H T_* - \mathscr{A} c_* - H'_D (\boldsymbol{u}_* - \boldsymbol{v}_*) \cdot \boldsymbol{n}, \qquad (7.17)$$

with

$$H'_D = \frac{(2\pi)^{1/2}}{\alpha c_0} \left[\frac{\mathscr{R}_g + (\mathscr{R}_v - \mathscr{R}_g) c_0}{\mathscr{R}_v} \right] = \frac{\rho_0}{\alpha \rho_{v0}} \left(\frac{2\pi p_0}{\rho_0 T_0 \mathscr{R}_v} \right)^{1/2}, \qquad (7.18)$$

and contains the saturation relation (7.15) as a special case.

7.1. *The local problem for diffusion*

We now pass to quantities made dimensionless according to the local scaling of §3.2 but, as before, omit any specific indication of this fact.

The hydrodynamic fields are identical to those found previously. Equations (5.12) for the drop surface temperature, (5.13) for the liquid temperature, and (5.14) for the surface-averaged heat flux also hold with the same expression (5.15) for the correction q to the continuum heat transfer coefficient. As a consequence, we again find the relation (5.16) between Q_0 and Θ_0. We write the surface vapour concentration in the form, reminiscent of (5.12),

$$c|_{r=1} = \bar{c} + C_0 + C_1 U \cdot n, \tag{7.19}$$

and the diffusion mass flux at the surface as

$$j \cdot n|_{r=1} = \frac{J_0}{4\pi} + J_1 U \cdot n, \tag{7.20}$$

analogous to (5.18). From (5.6) and (7.11) we then find

$$J_0 = \frac{1}{\Gamma} \dot{M}, \qquad J_1 = \frac{1}{\Gamma} \left(1 - \frac{3}{2A} + 2B \right). \tag{7.21}$$

The Hertz–Knudsen relation (7.17) now requires, in place of (5.17),

$$\Theta_0 = \overline{T}_{sat} - \overline{T} + \frac{H'_D}{4\pi H} \dot{M} + \frac{\mathscr{A}}{H} C_0, \qquad H\Theta_1 = \frac{H'_D}{H} \left(1 - \frac{3}{2A} + 2B \right) + \frac{\mathscr{A}}{H} C_1, \tag{7.22}$$

where

$$\overline{T}_{sat} = \frac{1}{H} (\bar{p} + \mathscr{A}\bar{c}) \tag{7.23}$$

is the saturation temperature.

Again according to Gyarmathy (1982), diffusional mass transfer rates from the drop at finite Knudsen numbers may be expressed in a form similar to (5.16), namely

$$J_0 = \frac{4\pi}{d(Kn)} C_0, \tag{7.24}$$

where

$$d(Kn) = 1 + 2.785 \frac{Kn}{Sc}. \tag{7.25}$$

From (7.21) we then have

$$\dot{M} = \frac{4\pi\Gamma}{d} C_0. \tag{7.26}$$

Upon substitution into the first of (7.22) and use of (5.16) and (5.19) we then find

$$\left. \begin{aligned} \dot{M} &= \frac{4\pi\Lambda}{q + (\Lambda/H)\left(H'_D + d\mathscr{A}/\Gamma\right)} \left(\overline{T} - \overline{T}_{sat} \right), \\ Q_0 &= -\frac{4\pi}{q + (\Lambda/H)\left(H'_D + d\mathscr{A}/\Gamma\right)} \left(\overline{T} - \overline{T}_{sat} \right). \end{aligned} \right\} \tag{7.27}$$

These results extend (5.21) to gas–vapour mixtures.

If Fick's law (7.14) applies, the local diffusion problem can be solved completely. This step is actually needed to enable us to calculate the continuum-value A^c of the drag constant A appearing in the drag correction (5.11). The calculation is very

similar to that at the end of §5 and one finds

$$\frac{1}{A^c} = 1 - \frac{\Lambda(K+2)}{2\,Re\,H + \Lambda(K+2)\left(1 + 2\,Re\,H'_D + Re\mathscr{A}/\Gamma\right)}. \tag{7.28}$$

Furthermore, from the second of (7.22), we have in this case

$$-\frac{6}{Q_1} = 2\,Re\,H + \Lambda(K+2)\left(1 + 2\,Re\,H'_D + \frac{Re\mathscr{A}}{\Gamma}\right). \tag{7.29}$$

7.2. *Effective equations*

The effective equations are readily derived. Like the case of the previous section, the dimensionless equation of continuity (6.13) acquires an additional contribution due to the $\nabla^2_* c_*$ in the left-hand side of (7.9) and is

$$\nabla\cdot\bar{\boldsymbol{u}} + \frac{1}{\hat{\gamma}}\frac{\partial\bar{p}}{\partial t} = \delta^2 n\left[1 + \frac{\epsilon}{\delta}\left(\frac{W}{Sc\Gamma} - \frac{1}{Pr\Lambda}\right)\right]\dot{M}. \tag{7.30}$$

The energy equation is formally identical to (6.14). The diffusion equation is

$$\frac{\partial\bar{c}}{\partial t} = \frac{n\delta\epsilon}{\Gamma Sc}\dot{M}. \tag{7.31}$$

As in the pure-vapour case of the previous section, upon combining these equations one can rewrite the equation of continuity in the form (6.15). Here $\dot{M}$ is given by the first of (7.27).

8. Summary and discussion

We have studied the problem of the propagation of linear pressure waves in a fog consisting of droplets suspended in a mixture of an incondensible gas and their own vapour. The scaling analysis of §3 shows that, for the present weak waves, phase change effects are only important when the Knudsen number Kn is of order 1, a condition that can be expressed as

$$a \sim \frac{\mu}{(p_0\rho_0)^{1/2}}, \tag{8.1}$$

or smaller. Here p_0, ρ_0, and μ are the undisturbed gas mixture pressure, density, and viscosity and a is the drop radius. This is the case that has been treated in detail for equal sized droplets. The extension to a droplet size distribution has been indicated at the end of §4. The deviation of the governing equations from the standard form applicable in the continuum limit $Kn \to 0$ has been accounted for by the use of approximate relations. The derivation, which is based on the smallness of the ratio $\delta = a/\lambda$, where λ is the typical wavelength of the perturbation, has been carried out in the time domain. The effective equations that we have derived are valid in the dilute limit where the droplet volume fraction is of the order of δ.

The dimensionless form of the gas-phase averaged momentum equation is given in (6.30). Upon returning to dimensional variables using (3.3), (3.5), (3.11), and $n_* = n_0/\lambda^3$, we find

$$\rho_0\frac{\partial\boldsymbol{u}}{\partial t} + \nabla p = \frac{6\pi}{A}\mu a n_0(x)(\boldsymbol{v} - \boldsymbol{u}). \tag{8.2}$$

Here and in the following the field quantities are to be interpreted in an average sense. The time derivative is partial rather than convective owing to linearization. The index

0 indicates equilibrium values. The droplet number per unit volume (not necessarily spatially uniform) is denoted by n_0, p is the pressure, and u, v are the gas and droplet velocity fields respectively. The drag correction A, accounting for the modification of the stress at the particle surface due to free-molecular and phase change effects, is given by (5.11) and (5.24) or (7.28).

A term $\nabla^2 u$ of the Brinkman type is absent from (8.2) because it is negligible in the present scaling in which the fundamental macroscopic length is the wavelength of the pressure perturbation.

The average equation of continuity is

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot u = n_0 \dot{M},$$ (8.3)

where $\dot{M}$ is the dimensional average evaporative mass flux per drop related to the corresponding dimensionless quantity $\dot{M}_*$ by

$$\dot{M}_* = \frac{\dot{M}}{\rho_0 M C a^2},$$ (8.4)

as follows from the non-dimensionalization of §3.

Similarly, with (3.5), (3.11), and (3.16), the dimensionless energy equation (6.14) is

$$\rho_0 C_p \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} = -n_0 L \dot{M},$$ (8.5)

where C_p is the constant-pressure specific heat of the gas–vapour mixture and L the latent heat, and the diffusion equation (7.31) is

$$\frac{\partial c}{\partial t} = \frac{\rho_{g0}}{\rho_0^2} n_0 \dot{M}.$$ (8.6)

Here c is the average vapour mass fraction and ρ_{g0} the undisturbed density of the incondensible gas.

The mean drop evaporation rate is given in dimensionless form in (7.27). Upon using (8.4) and the definitions (3.16), (3.30), (7.12), (7.16), and (7.18), the dimensional form of this result may be written

$$\dot{M} = \frac{4\pi a K}{L} \left[1 + 1.462 \frac{Kn}{Pr} + \frac{1}{\alpha} \frac{h_2}{a} + \left(1 + 2.785 \frac{Kn}{Sc} \right) \Phi \frac{p_{g0}}{p} \right]^{-1} [T - T_{sat}(p,c)],$$ (8.7)

where T_{sat} is the saturation temperature corresponding to p, c (see (7.15)), α is the accommodation coefficient (see (2.10)), the length h_2 is defined by

$$h_2 = \frac{k \mathscr{R}_v T_0^2}{L^2 \rho_{v0}} \left(\frac{2\pi}{\mathscr{R}_v T_0} \right)^{1/2},$$ (8.8)

and the quantity Φ by

$$\Phi = \frac{k \mathscr{R}_v T_0^2}{L^2 \rho_{v0} \mathscr{D}}.$$ (8.9)

If the terms in the first square brackets in (8.7) were replaced by 1, this expression for $\dot{M}$ would take on the form expected if the latent heat were provided by steady conduction over a distance a. For water h_2 has the value 1.5×10^{-5} μm at 20 °C, has a mimimum of 1.5×10^{-7} μm around 280 °C, and then increases very close to the critical point. This term is therefore unlikely to be important under normal circumstances.

For the quantity Φ, assuming the Schmidt number to be of order 1, we have $\Phi \approx 1.23 \times 10^{-4}$ and 0.28 at 20 °C and 300 °C, respectively. Depending on the vapour concentration, this term may therefore have a noticeable effect.

The dimensional form of the momentum equation (6.18) of the disperse phase is

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{9\mu}{2 A a^2 \rho_L} (\bar{\boldsymbol{u}} - \bar{\boldsymbol{v}}). \tag{8.10}$$

The corresponding energy equation is not necessary to close the system but is given in (6.19) for completeness.

The physical meaning of the effective equations derived is transparent. The vapour equation of conservation of mass acquires a source term given by the product of the net evaporation rate per drop multiplied by the drop number density. The energy needed to drive this source appears as a sink in the vapour energy equation. The equation for the concentration c is really a mass conservation equation for the vapour, and therefore it acquires the same source term suitably modified to appear as a source of mass fraction rather than of mass.

Even though the method that we have followed does not necessarily lead to linear equations (see e.g. Caflisch *et al.* 1985), the mathematical model that we have obtained is linear. This is due to the usually large value of the latent heat that limits the phase change rate at the drop surface. Nonlinear effects would however be important near the critical point as well as for stronger waves.

As noted at the beginning, the preceding system of average equations is valid when the drop radius is comparable with the molecular mean free path, see (8.1). For smaller droplets, phase change effects are even stronger than those considered here, but deviations from the continuum description are also greater and the model invalid. This case must be analysed on the basis of equations rooted in kinetic theory. For droplets larger than (8.1), on the other hand, the usual continuum relations are applicable, but the coupling between the phases is mainly through the interchange of momentum and therefore not different from that occurring in a dusty gas with no phase change. We thus conclude that the peculiarities of phase change need only be considered when the continuum flow model starts to break down.

It may be of some interest to compare (8.2), (8.3), (8.5) and (8.10) with other models in the literature. As a typical example, which contains most of the other ones proposed, we take that of Gumerov *et al.* (1988). After some adjustments to the notation to facilitate comparison with the present results, the vapour continuity equation of that model may be written

$$\frac{\partial}{\partial t}(\alpha_v \rho) + \alpha_{v0} \frac{\partial u}{\partial x} = n_0 \dot{M}, \tag{8.11}$$

where α_v is the vapour volume fraction. The vapour momentum equation is

$$\alpha_{v0} \rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = -n_0 f, \tag{8.12}$$

where f is the force on the droplets, and the energy equation is

$$\alpha_{v0} \rho_0 C_p \frac{\partial T}{\partial t} - \alpha_{v0} \frac{\partial p}{\partial x} = -n_0 q_{1\sigma}, \tag{8.13}$$

where $q_{1\sigma}$ is the heat flux from the vapour to the drop surface. Since, in the limit considered here, $\alpha_v \approx 1$, the left-hand sides of these equations coincide with those

of (8.3), (8.2), and (8.5). Differences arise however when the explicit expressions of the inter-phase coupling terms $\dot{M}$, f, and $q_{1\sigma}$ are examined. While, on the one hand, the expressions given by Gumerov *et al.* (1988) contain a large number of effects that our scaling analysis has proven to be small for linear waves, they do not include our finite-Kn corrections. Furthermore, the scaling analysis of §3 shows that some of the effects of Gumerov *et al.* (e.g. added mass force, sensible heat storage in the vapour, and others), cannot be significant unless $a \sim \lambda$, the drop volume fraction is not small, or nonlinearity is important. In this sense, the equations of Gumerov *et al.*, while based on a penetrating physical intuition, may be said to be inconsistent as they include some effects but omit others of a comparable magnitude.

Our equations have been *derived* rather than postulated. They are the simplest mathematical model containing all – and only – the physical effects that contribute in the limit of weak waves, far from the critical point, and for droplet radii comparable with the mean free path. In this sense, the most significant results of the paper are the explicit form of the corrections to the drag and mass flux terms. In particular, it is noteworthy that these terms are strongly affected by the finite-Kn corrections to the droplet heat flux and drag and by the vapour pressure–temperature relation at the interface. This remark suggests the possibility of exploring experimentally these problems uncluttered by the myriad of other effects that *a priori* may be expected to play a role. Conversely, our analysis indicates that the experimental investigation of effects that we have shown to be negligible in the case of weak waves would require data reduction on the basis of a more complex, nonlinear, non-dilute theoretical model.

As a final point, we have set out in §4 the details of a mathematical technique that appears widely applicable to the systematic derivation of effective equations for dilute multi-phase disperse flows. The strong aspect of this technique is its fundamental simplicity. On the negative side, it must be recognized that it does not appear to be readily susceptible of extension to larger volume fractions, although nonlinearity *per se* would not be an obstacle.

REFERENCES

AOKI, K. & CERCIGNANI, C. 1983 Evaporation and condensation on two parallel plates at finite Reynolds numbers. *Phys. Fluids* **26**, 1163–1164.

AOKI, K., SONE, Y. & YAMADA T. 1990 Numerical analysis of gas flows condensing on its plane condensed phase on the basis of kinetic theory. *Phys. Fluids* A **2**, 1867–1878.

CAFLISCH, R. E., MIKSIS, M. J., PAPANICOLAOU, G. C. & TING, L. 1985 Effective equations for wave propagation in bubbly liquids. *J. Fluid Mech.* **153**, 259–273.

CAFLISCH, R. E. & RUBINSTEIN, J. 1986 *Lectures on the Mathematical Theory of Multi-Phase Flow.* Lecture Notes, New York University, Courant Institute of Mathematical Sciences.

CERCIGNANI, C. 1988 *Kinetic Theory and Gas Dynamics.* Springer.

CERCIGNANI, C., FISZDON, W. & FREZZOTTI, A. 1985 The paradox of the inverted temperature profiles between an evaporating and a condensing surface. *Phys. Fluids* **28**, 3237–3240.

DURAISWAMI, R. 1990 I. Effective equations for sound propagation in fogs. II. Orthogonal mapping in two dimensions. PhD Dissertation, Department of Mechanical Engineering, The Johns Hopkins University (available from University Microfilms, Ann Arbor, Michigan).

FOLDY, L. L. 1945 The multiple scattering of waves. *Phys. Rev.* **67**, 107–119.

FUKUTA, N. & WALTER, L.A. 1970 Kinetics of hydrometer growth from a vapor spherical model. *J. Atmos. Sci.* **27**, 1160–1172.

GUHA, A. 1992 Structure of partly dispersed normal shock waves in vapor-droplet flows. *Phys. Fluids* A **4**, 1566–1578.

GUHA, A. 1994 A unified theory of aerodynamic and condensation shock waves in vapor-droplet flows with or without a carrier gas. *Phys. Fluids* **6**, 1893–1913.

GUMEROV, N. A., IVANDAEV, A. I. & NIGMATULIN, R.I. 1988 Sound waves in monodisperse gas-particle or vapour-droplet mixtures. *J. Fluid Mech.* **193**, 53–74.

GYARMATHY, G. 1982 The spherical droplet in gaseous carrier streams: Review and synthesis. In *Multiphase Science and Technology* (ed. G. F. Hewitt, J. M. Delhaye & N. Zuber), Vol. 1, pp. 99–279. Hemisphere.

HERMANS, L. J. F. & BEENAKKER, J. J. M. 1986 The temperature paradox in the kinetic theory of evaporation. *Phys. Fluids* **29**, 4231–4232.

JAESCHKE, M., HELLER, W. J. & MEIER, G.E.A. 1975 Acoustic damping in a gas mixture with suspended submicroscopic droplets. *J. Sound Vib.* **43**, 467–481.

KOFFMAN, L. D., PLESSET, M. S. & LEES, L. 1984 Theory of evaporation and condensation. *Phys. Fluids* **27**, 876–880.

KOGAN, M. N. 1973 Molecular gas dynamics. *Ann. Rev. Fluid Mech.* **5**, 383–404.

LABUNTSOV, D. A. & KRYUKOV, A. P. 1979 Analysis of intensive evaporation and condensation. *Intl J. Heat Mass Transfer* **22**, 989–1002.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics.* Pergamon.

LANDAU, L. D. & LIFSHITZ, E. M. 1981 *Physical Kinetics.* Pergamon.

MARBLE, F. E. 1969 Some gasdynamic problems in the flow of condensing vapors. *Astronautica Acta* **14**, 585–614.

MARBLE, F. E. & WOOTEN, D. C. 1970 Sound attenuation in a condensing vapor. *Phys. Fluids* **13**, 2657–2664.

MILLIKAN, R. A. 1923 The general law of fall of a small spherical body through a gas, and its bearing upon the nature of molecular reflection from surfaces. *Phys. Rev.* **22**, 1–23.

ONISHI, Y. 1986 The spherical-droplet problem of evaporation and condensation in a vapour-gas mixture. *J. Fluid Mech.* **163**, 171–194.

PAO, Y. P. 1971a Application of kinetic theory to the problem of evaporation and condensation. *Phys. Fluids* **14**, 306–312.

PAO, Y. P. 1971b Temperature and density jumps in the kinetic theory of gases and vapors. *Phys. Fluids* **14**, 1340–1346.

RUBINSTEIN, J. 1986 Effective equations for flow in random porous media with a large number of scales. *J. Fluid Mech.* **170**, 379–383.

SCHAAF, S. A. & CHAMBRÉ, P. L. 1961 *Flow of Rarefied Gases*, p. 39. Princeton University Press.

SHANKAR, P. N. & DESHPANDE, M. D. 1990 On the temperature distribution in liquid-vapor phase change between plane liquid surfaces. *Phys. Fluids* A **2**, 1030–1038.

SONE, Y., TAKATA, S., & WAKABAYASHI, M. 1994 Numerical analysis of a rarefied gas flow past a volatile particle using the Boltzmann equation for hard-sphere molecules. *Phys. Fluids* **6**, 1914–1928.

SUGIMOTO, H. & SONE, Y. 1992 Numerical analysis of steady flows of a gas evaporating from its cylindrical condensed phase on the basis of kinetic theory. *Phys. Fluids* A **4**, 419–440.

WEI, R., TIAN, Y. & LU, Q. 1987 Absorption of sound in water fog composed of submicron droplets. *J. Acoust. Soc. Am.* **81**, 1350–1354.

WEI, R. & WU, J. 1981 Absorption of sound in water fog. *J. Acoust. Soc. Am.* **70**, 1213–1219.

YOUNG, J. B. 1982 The spontaneous condensation of steam in supersonic nozzles. *PhysicoChem. Hydrodyn.* **3**, 57–82.

YOUNG, J. B. 1984 Semi-analytical techniques for investigating thermal non-equilibrium effects in wet steam turbines. *Intl J. Heat Fluid Flow* **5**, 81–91.

YOUNG, J. B. & GUHA, A. 1991 Normal shock wave structure in two-phase vapour-droplet flows. *J. Fluid Mech.* **228**, 243–274.

YTREHUS, T. 1983 Asymmetries in evaporation and condensation Knudsen layer problems. *Phys. Fluids* **26**, 939–952.